

# NILPOTENT TOEPLITZ OPERATORS ON REINHARDT DOMAINS

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**ABSTRACT.** We construct explicit examples of non-trivial nilpotent Toeplitz operators on Bergman spaces of certain Reinhardt domains in  $\mathbb{C}^2$ .

## 1. INTRODUCTION

**1.1. Set-up and Result.** Let  $\Omega$  be a domain in  $\mathbb{C}^n$  and  $A^2(\Omega)$  denote the Bergman space of  $\Omega$ . The Bergman projection operator,  $\mathcal{B}_\Omega$ , is the orthogonal projection from  $L^2(\Omega)$  onto  $A^2(\Omega)$ . It is an integral operator with the kernel called the Bergman kernel, denoted by  $B_\Omega(z, w)$ . If  $\{e_n(z)\}_{n=0}^\infty$  is an orthonormal basis for  $A^2(\Omega)$  then the Bergman kernel can be represented as  $B_\Omega(z, w) = \sum_{n=0}^\infty e_n(z) \overline{e_n(w)}$ . See [Kra01] for general theory of Bergman spaces.

For a function  $u$  on  $\Omega$ , the Toeplitz operator  $T_u : A^2(\Omega) \rightarrow A^2(\Omega)$  with the symbol  $u$  is defined by  $T_u(f) = \mathcal{B}_\Omega(uf)$ .

In this note, we are interested in the zero product problem. For two symbols  $u_1$  and  $u_2$ , if the product  $T_{u_1}T_{u_2}$  is identically zero on  $A^2(\Omega)$  then can we claim  $T_{u_1}$  or  $T_{u_2}$  is identically zero? This is a non-trivial problem and the answer is not even known when  $\Omega$  is the unit disc.

Here, we indicate the problem has a different flavor in higher dimensions. In particular, we present a family of Reinhardt domains in  $\mathbb{C}^2$  on which not only zero products of non-trivial Bergman space Toeplitz operators exist but we can find nilpotent Toeplitz operators.

**Theorem 1.** *There exists Reinhardt domains in  $\mathbb{C}^2$  on whose Bergman spaces there are nilpotent Toeplitz operators.*

*Remark 1.* It becomes clear in the proof that the operators in Theorem 1 are also of infinite rank.

**1.2. History:** The zero product problem on the Hardy space is initiated in [BH64]. It is completely solved in [AV09] where authors established that the product of non-zero Toeplitz operators is never zero. For the intermediate results, before the complete solution, see [Guo96, Gu00] and the references in [AV09].

In [AČ01a], it is shown that for the Toeplitz operators on the Bergman space  $A^2(\mathbb{D})$  of the unit disc  $\mathbb{D}$ , the analogue of the Brown-Halmos theorem holds under some additional hypothesis that  $u$  and  $v$  are bounded and harmonic. Later, the same result is proven for radial symbols in [AČ01b]. The problem on  $\mathbb{D}$ , without extra assumptions on the symbols, remains open.

The higher dimensional cases are studied in [CK06, CLNZ07, CKL07] where the results on the unit disc are extended to the ball or to the polydisk. In these papers, neither non-trivial zero products nor nilpotent Toeplitz operators are observed.

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In [BL11], the problem is considered on the Segal-Bargmann space (the space of square integrable entire functions on  $\mathbb{C}^n$  with a Gaussian decay weight) and an example of a non-trivial zero product of three Toeplitz operators is constructed. However, no nilpotent Toeplitz operator is observed.

## 2. PROOF OF THE THEOREM

Inspired by the construction in [Wie84], we define the following family of domains  $\Omega_m$  in  $\mathbb{C}^2$ .

$$\begin{aligned} X &= \left\{ (z_1, z_2) \in \mathbb{C}^2 : |z_1| > e, \quad |z_2| < \frac{1}{|z_1| \log |z_1|} \right\} \\ Y_m &= \left\{ (z_1, z_2) \in \mathbb{C}^2 : |z_2| > 2, \quad \left| |z_1| - \frac{1}{|z_2|} \right| < \frac{1}{|z_2|^m} \right\} \\ Z &= \left\{ (z_1, z_2) \in \mathbb{C}^2 : |z_1| \leq e, \quad |z_2| \leq 2 \right\} \end{aligned}$$

and put

$$\Omega_m = X \cup Y_m \cup Z, \quad m = 1, 2, \dots$$

Each  $\Omega_m$  is an unbounded Reinhardt domain with finite volume, see Figure 1.

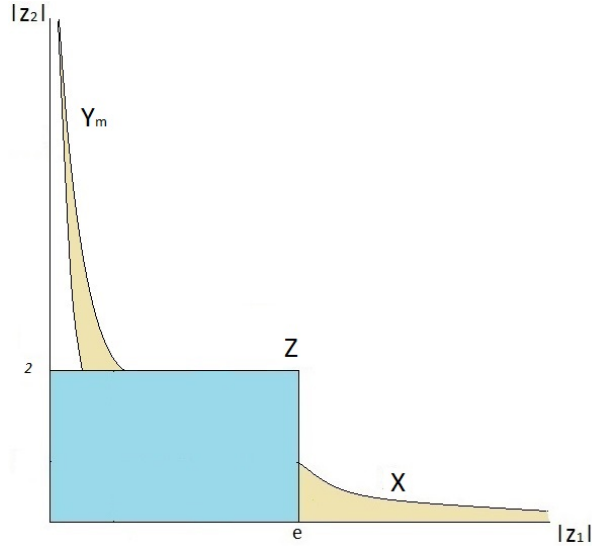


FIGURE 1. Representation of  $\Omega_m$  in absolute space  $\{(r_1, r_2) \in \mathbb{R}^2 \mid r_1 \geq 0 \text{ and } r_2 \geq 0\}$ , under the map  $\tau : (z_1, z_2) \rightarrow (|z_1|, |z_2|)$ .

**Lemma 1.** *For a multi-index  $\alpha = (\alpha_1, \alpha_2)$ , the monomial  $z^\alpha$  is in  $A^2(\Omega_m)$  if and only if  $\alpha_2 \geq \alpha_1 > \alpha_2 - \frac{m-1}{2}$ .*

*Proof.* We start with the calculation on the domain  $X$ .

$$\begin{aligned} \int_X |z^\alpha|^2 dV(z) &= \int_{|z_1| > e} |z_1|^{2\alpha_1} \int_{|z_2| < \frac{1}{|z_1| \log |z_1|}} |z_2|^{2\alpha_2} dA(z_2) dA(z_1) \\ &= 4\pi^2 \int_e^\infty r_1^{2\alpha_1+1} \int_0^{\frac{1}{r_1 \log(r_1)}} r_2^{2\alpha_2+1} dr_2 dr_1 = \frac{4\pi^2}{2\alpha_2+2} \int_e^\infty \frac{r_1^{2\alpha_1+1}}{r_1^{2\alpha_2+2} (\log(r_1))^{2\alpha_2+2}} dr_1. \end{aligned}$$

We note that for  $k > 0$ , the improper integral  $\int_e^\infty \frac{1}{x^m (\log x)^k} dx$  converges if and only if  $m \geq 1$ . Therefore, the last integral above (where  $k = 2\alpha_2 + 2 > 0$  and  $m = 2(\alpha_2 - \alpha_1) + 1$ ) is finite if and only if  $(\alpha_2 - \alpha_1) \geq 0$ . In other words

$$(1) \quad z^\alpha \in A^2(X) \Leftrightarrow \alpha_2 \geq \alpha_1.$$

We continue with the calculation on the domain  $Y_m$ .

$$\begin{aligned} \int_{Y_m} |z^\alpha|^2 dV(z) &= \int_{|z_2| > 2} |z_2|^{2\alpha_2} \int_{\frac{1}{|z_2|} - \frac{1}{|z_2|^m} < |z_1| < \frac{1}{|z_2|} + \frac{1}{|z_2|^m}} |z_1|^{2\alpha_1} dA(z_2) dA(z_1) \\ &= 4\pi^2 \int_2^\infty r_2^{2\alpha_2+1} \int_{\frac{1}{r_2} - \frac{1}{r_2^m}}^{\frac{1}{r_2} + \frac{1}{r_2^m}} r_1^{2\alpha_1+1} dr_1 dr_2 \\ &= \frac{4\pi^2}{2\alpha_1+2} \int_2^\infty r_2^{2\alpha_2+1} \left[ \left( \frac{1}{r_2} - \frac{1}{r_2^m} \right)^{2\alpha_1+2} - \left( \frac{1}{r_2} + \frac{1}{r_2^m} \right)^{2\alpha_1+2} \right] dr_2. \end{aligned}$$

Since  $r_2 > 2$ , after using the binomial expansion in the brackets we consider the term  $1/r_2$  with the smallest degree as the dominant, which is  $1/r_2^{2\alpha_1+1+m}$ . The last integral can be estimated by

$$\int_{Y_m} |z^\alpha|^2 dV(z) \approx \int_2^\infty r_2^{2\alpha_2+1} \frac{1}{r_2^{2\alpha_1+1+m}} dr_2.$$

The integral on the right is finite if and only if  $\alpha_1 > \alpha_2 + \frac{1-m}{2}$ . In other words

$$(2) \quad z^\alpha \in A^2(Y_m) \Leftrightarrow \alpha_1 > \alpha_2 + \frac{1-m}{2}.$$

The lemma follows from (1) and (2). □

Next, we set  $m \geq 6$ ,  $\phi = z_1/\bar{z}_1$  and consider  $T_\phi$  on  $A^2(\Omega_m)$ .

**Proposition 1.** *The following properties hold.*

- (i)  $T_\phi$  is not a zero operator.
- (ii)  $T_\phi$  does not have finite rank.
- (iii)  $T_\phi$  is a bounded operator.
- (iv)  $T_\phi$  is a nilpotent operator of degree  $\lfloor \frac{m}{4} \rfloor$ , the largest integer less than or equal to  $\frac{m}{4}$ .

*Remark 2.* Once we prove Proposition 1, we immediately obtain Theorem 1. However, it will be clear in the proof that the domain and the operator we present aren't unique but part of a family of domains and operators. We leave exploration of more examples to the reader.

Before we start proving Proposition 1, we define the following lattice for  $m \geq 6$ .

$$\begin{aligned} R_m &= \left\{ (\alpha_1, \alpha_2) \in \mathbb{N}^2 \mid \alpha_2 \geq \alpha_1 > \alpha_2 - \frac{m-1}{2} \right\} \\ &= \left\{ (\alpha_1, \alpha_2) \in \mathbb{N}^2 \mid \alpha_1 + \frac{m-1}{2} > \alpha_2 \geq \alpha_1 \right\}. \end{aligned}$$

*Remark 3.* Shifting  $\alpha_1$  to the right by a number  $s$  greater than or equal to  $\frac{m-1}{2}$  is enough to put the resulting index  $(\alpha_1 + s, \alpha_2)$  out of  $R_m$ . That is, if  $(\alpha_1, \alpha_2) \in R_m$  then for  $s \geq \frac{m-1}{2}$  we get  $(\alpha_1 + s, \alpha_2) \notin R_m$ .

For a multi-index  $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{N}^n$ , we set  $c_\gamma^2 = \int_\Omega |z^\gamma|^2 dV(z)$ . Then, on a radially symmetric domain  $\Omega$  that contains the origin, the set (or a subset of)  $\left\{ \frac{z^\gamma}{c_\gamma} \right\}_{\gamma \in \mathbb{N}^n}$  gives a complete orthonormal basis for  $A^2(\Omega)$ . Each  $f \in A^2(\Omega)$  can be written in the form  $f(z) = \sum_\gamma f_\gamma \frac{z^\gamma}{c_\gamma}$  where the sum converges in  $A^2(\Omega)$ , but also uniformly on compact subset of  $\Omega$ . For the coefficients  $f_\gamma$ , we have  $f_\gamma = \left\langle f(z), \frac{z^\gamma}{c_\gamma} \right\rangle_\Omega$ .

*Proof of Proposition 1.* Consider  $T_\phi$  on  $A^2(\Omega_m)$  for  $m \geq 6$ .  $\Omega_m$  is a radially symmetric domain and the monomials with exponents that reside in  $R_m$  form a complete system for  $A^2(\Omega_m)$ . By using the orthogonality of monomials we get

$$\begin{aligned} (3) \quad T_\phi(z^\alpha) &= \mathcal{B}_{\Omega_m} \left( \frac{z_1}{\bar{z}_1} \cdot z^\alpha \right) = \sum_{\gamma \in R_m} \left\langle \frac{z_1}{\bar{z}_1} z^\alpha, \frac{z^\gamma}{c_\gamma} \right\rangle \frac{z^\gamma}{c_\gamma} \\ &= \frac{c_{(\alpha_1+1, \alpha_2)}^2}{c_{(\alpha_1+2, \alpha_2)}^2} z_1^{\alpha_1+2} z_2^{\alpha_2}. \end{aligned}$$

On the above summation only  $(\gamma_1, \gamma_2) = (\alpha_1 + 2, \alpha_2)$  survives. Moreover, there exists multi-indices  $(\alpha_1, \alpha_2)$  in  $R_m$  such that  $(\alpha_1 + 2, \alpha_2)$  is also in  $R_m$ . Therefore, there exists  $z^\alpha \in A^2(\Omega_m)$  such that  $T_\phi(z^\alpha) = \frac{c_{(\alpha_1+1, \alpha_2)}^2}{c_{(\alpha_1+2, \alpha_2)}^2} z_1^{\alpha_1+2} z_2^{\alpha_2} \in A^2(\Omega_m)$  and  $T_\phi$  is a non-zero operator.

For  $m \geq 6$  and  $k \in \mathbb{N}$ ,  $z_1^k z_2^{k+2} \in A^2(\Omega_m)$  and  $T_\phi \left( z_1^k z_2^{k+2} \right) = \frac{c_{(k+1, k+2)}^2}{c_{(k+2, k+2)}^2} z_1^{k+2} z_2^{k+2} \in A^2(\Omega_m)$ .

Hence, the range of the operator  $T_\phi$  contains all the monomials of the form  $z_1^{k+2} z_2^{k+2}$  and so the range of  $T_\phi$  is infinite dimensional.

If  $g(z_1, z_2) \in A^2(\Omega_m)$  then its series expansion will be

$$g(z_1, z_2) = \sum_{\alpha_1=0}^{\infty} \sum_{\alpha_2=\alpha_1}^{\alpha_1+r-1} g_{\alpha_1 \alpha_2} \frac{z_2^{\alpha_2} z_1^{\alpha_1}}{c_{(\alpha_1, \alpha_2)}} = \sum_{\alpha_1=0}^{\infty} \sum_{\alpha_2=\alpha_1}^{\alpha_1+r-1} \left\langle g(z), \frac{z^\alpha}{c_\alpha} \right\rangle \frac{z^\alpha}{c_\alpha},$$

where

$$r = \begin{cases} \frac{m}{2}, & \text{if } m \text{ is even} \\ \frac{m-1}{2}, & \text{if } m \text{ is odd} \end{cases}.$$

The norm of  $g(z_1, z_2)$  is given by

$$(4) \quad \|g\|_{A^2(\Omega_m)}^2 = \sum_{\alpha_1=0}^{\infty} \sum_{\alpha_2=\alpha_1}^{\alpha_1+r-1} |g_{\alpha_1\alpha_2}|^2$$

and the norm of  $T_\phi(g)$  is

$$(5) \quad \begin{aligned} \|T_\phi(g)\|_{A^2(\Omega_m)}^2 &= \left\| \sum_{\alpha_1=0}^{\infty} \sum_{\alpha_2=\alpha_1}^{\alpha_1+r-1} \left\langle \frac{z_1}{\bar{z}_1} \cdot g(z), \frac{z^\alpha}{c_\alpha} \right\rangle \frac{z^\alpha}{c_\alpha} \right\|^2 \\ &= \sum_{\alpha_1=0}^{\infty} \sum_{\alpha_2=\alpha_1}^{\alpha_1+r-1} \left| \left\langle \frac{z_1}{\bar{z}_1} \cdot g(z), \frac{z^\alpha}{c_\alpha} \right\rangle \right|^2 = \sum_{\alpha_1=0}^{\infty} \sum_{\alpha_2=\alpha_1}^{\alpha_1+r-1} \left| \left\langle \frac{z_1}{\bar{z}_1} \cdot \sum_{\beta} g_{\beta} \frac{z^\beta}{c_\beta}, \frac{z^\alpha}{c_\alpha} \right\rangle \right|^2 \\ &= \sum_{\alpha_1=2}^{\infty} \sum_{\alpha_2=\alpha_1}^{\alpha_1+r-1} \left| \left\langle \frac{z_1}{\bar{z}_1} \cdot g_{(\alpha_1-2, \alpha_2)} z_1^{\alpha_1-2} z_2^{\alpha_2}, \frac{z^\alpha}{c_\alpha} \right\rangle \right|^2 \quad \text{by orthogonality of monomials} \\ &= \sum_{\alpha_1=2}^{\infty} \sum_{\alpha_2=\alpha_1}^{\alpha_1+r-1} \left| g_{(\alpha_1-2, \alpha_2)} \frac{c_{(\alpha_1-1, \alpha_2)}}{c_\alpha} \right|^2 \quad \text{then we shift the indices} \\ &= \sum_{\alpha_1=0}^{\infty} \sum_{\alpha_2=\alpha_1}^{\alpha_1+r-1} |\tilde{g}_{\alpha_1\alpha_2}|^2, \end{aligned}$$

where

$$\tilde{g}_{\alpha_1\alpha_2} = \begin{cases} 0, & \text{if } \alpha_1 = \alpha_2 \text{ or } \alpha_1 = \alpha_2 + 1 \\ \frac{c_{(\alpha_1+1, \alpha_2)}}{c_{(\alpha_1+2, \alpha_2)}} g_{\alpha_1\alpha_2}, & \text{otherwise} \end{cases}.$$

The ratio  $\frac{c_{(\alpha_1+1, \alpha_2)}^2}{c_{(\alpha_1+2, \alpha_2)}^2}$  is uniformly bounded by a constant. Indeed, each integral on  $X$  and  $Y_m$  has a uniform bound from above (say  $C_X$  and  $C_{Y_m}$ ) because of the conditions (1) and (2). Furthermore, we compute the integrals on the polydisc  $Z$  explicitly and estimate as follows

$$(6) \quad \frac{c_{(\alpha_1+1, \alpha_2)}^2}{c_{(\alpha_1+2, \alpha_2)}^2} \leq \frac{C_X + C_{Y_m} + \pi \frac{e^{2\alpha_1+4}}{\alpha_1+2} \cdot \pi \frac{2^{2\alpha_2+2}}{\alpha_2+1}}{\pi \frac{e^{2\alpha_1+2}}{\alpha_1+1} \cdot \pi \frac{2^{2\alpha_2+2}}{\alpha_2+1}} \leq \frac{C_X + C_{Y_m}}{\pi^2} + e^2 = C.$$

This estimate implies

$$(7) \quad |\tilde{g}_{\alpha_1\alpha_2}|^2 \leq C \cdot |g_{\alpha_1\alpha_2}|^2, \quad \text{for all } (\alpha_1, \alpha_2) \in R_m.$$

Thus, from (4), (5), and (7) it follows that

$$\|T_\phi(g)\|_{A^2(\Omega_m)}^2 \leq C \cdot \|g\|_{A^2(\Omega_m)}^2.$$

Finally, we calculate the powers of  $T_\phi$ .

$$\begin{aligned}
 (8) \quad T_\phi^2(z^\alpha) &= T_\phi \cdot T_\phi(z^\alpha) = T_\phi \left( \frac{c_{(\alpha_1+1, \alpha_2)}^2}{c_{(\alpha_1+2, \alpha_2)}^2} z_1^{\alpha_1+2} z_2^{\alpha_2} \right) \\
 &= \frac{c_{(\alpha_1+1, \alpha_2)}^2}{c_{(\alpha_1+2, \alpha_2)}^2} \cdot \frac{c_{(\alpha_1+3, \alpha_2)}^2}{c_{(\alpha_1+4, \alpha_2)}^2} z_1^{\alpha_1+4} z_2^{\alpha_2}.
 \end{aligned}$$

As for the third power,

$$(9) \quad T_\phi^3(z^\alpha) = \frac{c_{(\alpha_1+1, \alpha_2)}^2}{c_{(\alpha_1+2, \alpha_2)}^2} \cdot \frac{c_{(\alpha_1+3, \alpha_2)}^2}{c_{(\alpha_1+4, \alpha_2)}^2} \cdot \frac{c_{(\alpha_1+5, \alpha_2)}^2}{c_{(\alpha_1+6, \alpha_2)}^2} z_1^{\alpha_1+6} z_2^{\alpha_2}.$$

Continuing in that fashion, the  $k^{\text{th}}$  power of the operator is

$$(10) \quad T_\phi^k(z^\alpha) = \frac{c_{(\alpha_1+1, \alpha_2)}^2}{c_{(\alpha_1+2, \alpha_2)}^2} \cdot \frac{c_{(\alpha_1+3, \alpha_2)}^2}{c_{(\alpha_1+4, \alpha_2)}^2} \cdots \frac{c_{(\alpha_1+2k-1, \alpha_2)}^2}{c_{(\alpha_1+2k, \alpha_2)}^2} z_1^{\alpha_1+2k} z_2^{\alpha_2}.$$

In (10), if  $2k < r$  then there exists  $(\alpha_1, \alpha_2) \in R_m$  such that  $(\alpha_1 + 2k, \alpha_2) \in R_m$ , see the discussion in Remark 3, so  $z_1^{\alpha_1+2k} z_2^{\alpha_2} \in A^2(\Omega_m)$  and  $T_\phi^k \neq 0$  on  $A^2(\Omega_m)$ .

However, in (10), if  $2k \geq r$  then for all  $(\alpha_1, \alpha_2) \in R_m$  we have  $(\alpha_1 + 2k, \alpha_2) \notin R_m$  by Remark 3, so we see that  $z_1^{\alpha_1+2k} z_2^{\alpha_2} \notin A^2(\Omega_m)$  and  $T_\phi^k \equiv 0$  on  $A^2(\Omega_m)$ . That is,  $T_\phi$  is a nilpotent operator of degree  $k$  on  $A^2(\Omega_m)$ . □

We illustrate the main arguments of the proof in the following example.

*Example 1.* Set  $m = 9$ , then the monomial  $z_1^{\alpha_1} z_2^{\alpha_2}$  is in  $A^2(\Omega_9)$  if and only if  $\alpha_1 + 4 > \alpha_2 \geq \alpha_1$ . The exponents of the monomial in  $A^2(\Omega_9)$  are marked on the lattice below in Figure 2.

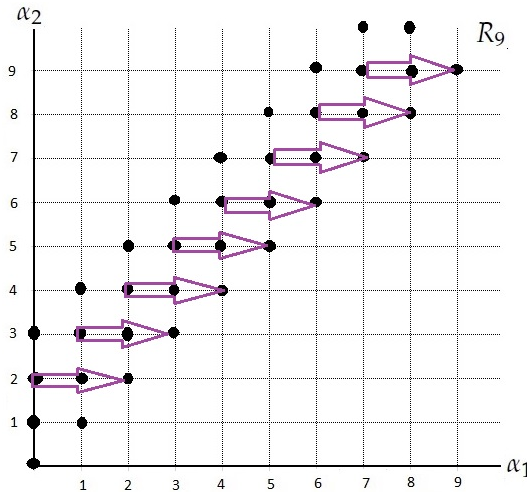


FIGURE 2. Representation of the lattice  $R_m$  for  $m = 9$  and the action of  $T_\phi$  on  $R_m$ .

It can be noted that  $T_\phi$  acts like a shift on the lattice, it takes  $(\alpha_1, \alpha_2 + 2)$  to  $(\alpha_1 + 2, \alpha_2 + 2)$ . Thus, if  $T_\phi$  is applied on any monomial two times then the exponent of the monomial runs out of the lattice  $R_9$ . That is, if  $z_1^{\alpha_1} z_2^{\alpha_2} \in A^2(\Omega_9)$  then  $T_\phi \cdot T_\phi(z_1^{\alpha_1} z_2^{\alpha_2}) = \frac{c_{(\alpha_1+1, \alpha_2)}^2}{c_{(\alpha_1+2, \alpha_2)}^2} \cdot \frac{c_{(\alpha_1+3, \alpha_2)}^2}{c_{(\alpha_1+4, \alpha_2)}^2} z_1^{\alpha_1+4} z_2^{\alpha_2} \notin A^2(\Omega_9)$  and so  $T_\phi^2 \equiv 0$  on  $A^2(\Omega_9)$ .

### 3. ACKNOWLEDGMENT

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